

Where Did Mally Go Wrong?

Gert-Jan C. Lokhorst

Section of Philosophy
Faculty of Technology, Policy and Management
Delft University of Technology, P.O. Box 5015,
2600 GA Delft, The Netherlands
g.j.c.lokhorst@tudelft.nl

Abstract. In 1926, Ernst Mally proposed the first system of deontic logic. His system turned out to be unacceptable. How can it be repaired? We discuss several proposals to reformulate it in terms of strict implication, relevant implication and strict relevant implication.

1 Introduction

In 1926, Ernst Mally proposed the first system of deontic logic [14]. He adopted the following principles:

1. If A requires B and if B then C , then A requires C .
2. If A requires B and if A requires C , then A requires B and C .
3. A requires B if and only if it is obligatory that if A then B .
4. The unconditionally obligatory is obligatory.
5. The unconditionally obligatory does not require its own negation.

Mally formalized these principles as follows. He used the language of the classical propositional calculus enriched with the connective O (“it is obligatory that”) and the propositional constant u (“the unconditionally obligatory,” *das unbedingt Geforderte*). He formalized “ A requires B ” as $A \supset OB$.

D1 $((A \supset OB) \& (B \supset C)) \supset (A \supset OC)$

D2 $((A \supset OB) \& (A \supset OC)) \supset (A \supset O(B \& C))$

D3a $(A \supset OB) \supset O(A \supset B)$

D3b $O(A \supset B) \supset (A \supset OB)$

D4 Ou

D5 $\sim(u \supset O\sim u)$

Mally’s deontic logic consisted of the classical propositional calculus plus axioms D1–D5. (Mally wrote D4 as $\exists uOu$, but this should be regarded as a mistake; see [6, pp. 2–3].)

Menger [15] showed that $A \equiv OA$ is a theorem of this system (Mally did not mention this formula), from which he concluded that it is unacceptable:

This result seems to me to be detrimental for Mally’s theory, however. It indicates that the introduction of the sign O is superfluous in the sense that it may be cancelled or inserted in any formula at any place

we please. But this result (in spite of Mally's philosophical justification) clearly contradicts not only our use of the word "ought" but also some of Mally's own correct remarks about this concept, e.g., the one at the beginning of his development to the effect that $A \supset (OB \vee OC)$ and $A \supset O(B \vee C)$ are not equivalent. Mally is quite right that these two propositions are not equivalent according to the ordinary use of the word "ought." But they are equivalent according to his theory by virtue of the equivalence of A and OA ([15, p. 58], notation adapted).

Menger's criticism is puzzling. The claim that "the sign O is superfluous in the sense that it may be cancelled or inserted in any formula at any place we please" is false because $A \vee B$ is well-formed but $AO \vee B$ is not. We conjecture that Menger intended to make the following objection:

Observation 1. *Mally's system is deontically trivial in the sense that it is closed under the following two rules:*

1. *from F to infer F^A/OA ,*
2. *from F to infer F^{OA}/A ,*

where F^A/B is the result of replacing one occurrence of A in F by B .

Mally's system is deontically trivial in this sense because it has the theorem $A \equiv OA$, as Menger showed, from which these two rules follow.

Where did Mally go wrong? How can Menger's and Mally's objections be avoided? We shall discuss several proposals. First, we shall show that reformulating it in terms of one of the Lewis systems of strict implication does not work: if all occurrences of material implication are replaced by strict implication, the resulting system is still objectionable for the reason pointed out by Menger. We do not exclude that one of the weaker Lemmon systems of strict implication might work (Section 2).

Second, we shall show that reformulating Mally's principles in terms of some system of relevant implication does lead to the desired result. This was already pointed out in [11] and [12]. We supplement this earlier proposal by the observation that choosing system **R** results in a deontic system that is weaker than necessary because it does not provide the relatively uncontroversial principle that obligation implies permission (where "it is permitted that A " is defined as "it is not obligatory that not A "). System **KR** of relevant logic (also known as "classical," "super-classical," "Boolean" or "super-Boolean" relevant logic) is better in this regard (Section 3).

Thirdly, we shall point out that the resulting system is unsatisfactory from a modern point of view. However, it can be reformulated in such a way that only one axiom needs modification to make it acceptable. This modification is similar to a modification suggested by Føllesdal and Hilpinen [6] (Section 4).

2 Strict Implication

We first study what happens if Mally's principles are reformulated in terms of strict implication. If one uses one of the Lewis systems, such as **S1**^o, **S1**, **S2**, **S3**,

S4 or **S5** (see [20]), then such a reformulation does not work, as we shall now show.

System **S** is formulated in the language of the propositional calculus with an additional unary connective \Box , read as “necessarily,” and the definitions $A \rightarrow B =_{\text{df}} \Box(A \supset B)$ and $A \varepsilon\rightarrow B =_{\text{df}} (A \rightarrow B) \& (B \rightarrow A)$. \rightarrow is read as “strictly implies.” $\varepsilon\rightarrow$ is read as “is strictly equivalent with.”

Definition 1. *S* has the following axioms and rules (named as in [4]):

PL *The set of all tautologies.*

\Box **PL** $\{\Box A: A \in \text{PL}\}$.

US *Uniform substitution.*

RRSE_T *From $A \varepsilon\rightarrow B$ and F to infer F^A/B .*

MP *From A and $A \supset B$ to infer B .*

SMP *From A and $A \rightarrow B$ to infer B .*

Definition 2. *If A is a formula of Mally’s deontic logic, then the strict version of A is the formula A^{\rightarrow} got by replacing all occurrences of \supset in A by \rightarrow .*

Observation 2. *S* plus the strict versions of Mally’s axioms $D3a$, $D3b$ and $D5$ is deontically trivial in the sense of Observation 1.

*Proof*¹

1	$(A \& \sim A) \rightarrow O\sim(A \& \sim A)$	\Box PL
2	$O((A \& \sim A) \rightarrow \sim(A \& \sim A))$	1 $D3a^{\rightarrow}$ SMP
3	$((A \& \sim A) \supset \sim(A \& \sim A)) \varepsilon\rightarrow (A \supset A)$	\Box PL PL
4	$O(A \rightarrow A)$	2 3 RRSE _T
5	$A \rightarrow OA$	4 $D3b^{\rightarrow}$ SMP
6	$((A \& \sim B) \supset C) \varepsilon\rightarrow ((A \& \sim C) \supset B)$	\Box PL PL
7	$(OA \& \sim B) \rightarrow OA$	\Box PL
8	$O((OA \& \sim B) \rightarrow A)$	7 $D3a^{\rightarrow}$ SMP
9	$O((OA \& \sim A) \rightarrow B)$	6 8 RRSE _T
10	$(OA \& \sim A) \rightarrow OB$	9 $D3b^{\rightarrow}$ SMP
11	$(OA \& \sim OB) \rightarrow A$	6 10 RRSE _T
12	$OA \rightarrow OA$	\Box PL
13	$O(OA \rightarrow A)$	12 $D3a^{\rightarrow}$ SMP
14	$\sim O(\mathbf{u} \rightarrow \sim \mathbf{u})$	$D3b^{\rightarrow}$ $D5^{\rightarrow}$ SMP
15	$O(OA \rightarrow A) \& \sim O(\mathbf{u} \rightarrow \sim \mathbf{u})$	13 14 PL
16	$OA \rightarrow A$	11 15 SMP
17	$A \varepsilon\rightarrow OA$	5 16 PL
18	$OA \varepsilon\rightarrow A$	16 5 PL
19	$F/F^A/OA$	17 RRSE _T
20	$F/F^{OA}/A$	18 RRSE _T

□

¹ This proof was inspired by the machine-generated derivation given in the Appendix.

S0.9° is axiomatized as **S** plus axiom $(A \rightarrow B) \rightarrow (\Box A \supset \Box B)$ (see [4]). **S** is contained in **S0.9°**, **S0.9**, **S1°**, **S1** and all the Lewis systems mentioned in [20], so the suggestion to reformulate Mally's assumptions in terms of one of these systems so as to avoid Menger's objection does not work.

On the other hand, some weaker system of strict implication might conceivably work, such as Lemmon's **S0.5** (which lacks RRSE_T) or his **E**-systems, which have no theorem of the form $\Box A$ (see [8]). Since the proof of Observation 2 given above depends on RRSE_T and $\Box\text{PL}$, it does not go through in these weaker systems. So even though not *all* kinds of strict implication are suitable, *some* of them might be. The resulting systems would be so weak, however, that it is to be feared that almost all of the strict versions of theorems mentioned by Mally (cf. Observation 4 below) would share the fate of $A \rightarrow OA$ and $OA \rightarrow A$ and fail to be derivable.

3 Relevant Implication

Let us explore an alternative suggestion and study the consequences of formulating Mally's assumptions in terms of *relevant* rather than material or strict implication (see [11] and [12]). There is a great variety of systems of relevance logic. We confine our attention to **R** and **KR** (see [1] and [2]). \rightarrow is the symbol for relevant implication. Relevant equivalence \leftrightarrow is defined by $A \leftrightarrow B =_{\text{df}} (A \rightarrow B) \& (B \rightarrow A)$.

Definition 3. *System **R** is axiomatized as follows:*

- R1** $A \rightarrow A$
- R2** $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
- R3** $A \rightarrow ((A \rightarrow B) \rightarrow B)$
- R4** $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
- R5** $(A \& B) \rightarrow A$
- R6** $(A \& B) \rightarrow B$
- R7** $((A \rightarrow B) \& (A \rightarrow C)) \rightarrow (A \rightarrow (B \& C))$
- R8** $A \rightarrow (A \vee B)$
- R9** $B \rightarrow (A \vee B)$
- R10** $((A \rightarrow C) \& (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$
- R11** $(A \& (B \vee C)) \rightarrow ((A \& B) \vee C)$
- R12** $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$
- R13** $\sim\sim A \rightarrow A$
- R14** $A \leftrightarrow (t \rightarrow A)$
- R15** $\sim A \leftrightarrow (A \rightarrow f)$
- R16** $A \rightarrow T$
- R17** $F \rightarrow A$
- $\rightarrow\mathbf{E}$ *From A and $A \rightarrow B$ to infer B*
- $\&\mathbf{I}$ *From A and B to infer $A \& B$.*

t is the conjunction of all provable statements, f is its negation. T is the *Verum* (which Mally wrote as V), F is the *Falsum* (which Mally wrote as Λ).

Definition 4. System \mathbf{KR} is \mathbf{R} plus $(A \& \sim A) \rightarrow B$.

Definition 5. If A is a formula of Mally's deontic logic, then the relevant version of A is the formula A^\rightarrow got by replacing all occurrences of \supset , \vee and \wedge by \rightarrow , \mathbf{T} and \mathbf{F} , respectively.

Definition 6. Given relevant system \mathbf{X} , \mathbf{X}^δ is \mathbf{X} plus the relevant versions of Mally's axioms D1–D5.

Definition 7. Given system \mathbf{X}^δ , $\mathbf{X}^{\delta+}$ is \mathbf{X}^δ plus $\mathbf{T} \rightarrow \mathbf{u}$.

Adding $\mathbf{T} \rightarrow \mathbf{u}$ leads to a better result in Observation 4 below.

Observation 3. $\mathbf{R}^{\delta+}$ and $\mathbf{KR}^{\delta+}$ do not provide the following formulas:

1. $A \rightarrow OA$
2. $OA \rightarrow A$
3. $O(A \vee B) \rightarrow (OA \vee OB)$

Proof. By the computer program MaGIC [19], which generates the following tables and lattice, in which the designated values are preceded by a *.

	\sim	O	\rightarrow	0	1	2	3
0	3	0	0	3	3	3	3
*1	2	0	1	0	1	2	3
2	1	1	2	0	0	1	3
*3	0	3	3	0	0	0	3

$\&$ and \vee are interpreted as lattice meet (greatest lower bound) and join (least upper bound), respectively. $\mathbf{F} = 0$, $\mathbf{t} = 1$, $\mathbf{f} = 2$, $\mathbf{u} = \mathbf{T} = 3$. All theorems of $\mathbf{KR}^{\delta+}$ receive the value 1 or 3, but $1 \rightarrow O1 = 0$, $O2 \rightarrow 2 = 2$ and $O(1 \vee 2) \rightarrow (O1 \vee O2) = 0$. □

$\mathbf{R}^{\delta+}$ and $\mathbf{KR}^{\delta+}$ therefore avoid both deontic triviality and the principle of distribution of obligation over disjunction, which Mally rejected (see the quote from Menger in Section 1 above, as well as [14, ch. 2, §4, p. 27, ad 3]).

Observation 4. The relevant versions of 39 of the 47 theorems discussed by Mally (83%) are derivable in $\mathbf{R}^{\delta+}$. The relevant versions of the other 8 theorems (17%) are not derivable in $\mathbf{R}^{\delta+}$. $\mathbf{KR}^{\delta+}$ does not differ from $\mathbf{R}^{\delta+}$ in this respect.

Proof. We follow the useful list given in [16, pp. 121–3]. A + indicates that the relevant version of Mally's theorem is a theorem of $\mathbf{R}^{\delta+}$, a – that it is not.

T1	(A → OB) → (A → OT)	+
T2	(A → OF) ↔ (∀M)(A → OM)	+
T2'	(A → OF) → (∀M)(A → OM)	+
T2''	(∀M)(A → OM) → (A → OF)	+

T3	$((A \rightarrow OB) \vee (A \rightarrow OC)) \rightarrow (A \rightarrow O(B \vee C))$	+
T4	$((A \rightarrow OB) \& (C \rightarrow OD)) \rightarrow ((A \& C) \rightarrow O(B \& D))$	+
T5a	$OA \leftrightarrow (\forall M)(M \rightarrow OA)$	-
T5b	$(\forall M)(M \rightarrow OA) \leftrightarrow (\forall M)(M \rightarrow OA)$	+
T6	$(OA \& (A \rightarrow B)) \rightarrow OB$	+
T7	$OA \rightarrow OT$	+
T8	$((A \rightarrow OB) \& (B \rightarrow OC)) \rightarrow (A \rightarrow OC)$	+
T9	$(OA \& (A \rightarrow OB)) \rightarrow OB$	+
T10	$(OA \& OB) \leftrightarrow O(A \& B)$	+
T11	$((A \rightarrow OB) \& (B \rightarrow OA)) \leftrightarrow O(A \leftrightarrow B)$	+
T12a	$(A \rightarrow OB) \leftrightarrow (A \rightarrow OB)$	+
T12b	$(A \rightarrow OB) \leftrightarrow O(A \rightarrow B)$	+
T12c	$O(A \rightarrow B) \leftrightarrow O\sim(A \& \sim B)$	-
T12d	$O\sim(A \& \sim B) \leftrightarrow O(\sim A \vee B)$	+
T13a	$(A \rightarrow OB) \leftrightarrow \sim(A \& \sim OB)$	-
T13b	$\sim(A \& \sim OB) \leftrightarrow (\sim A \vee OB)$	+
T14	$(A \rightarrow OB) \leftrightarrow (\sim B \rightarrow O\sim A)$	+
T15	$(\forall M)(M \rightarrow Ou)$	+
T16	$(u \rightarrow A) \rightarrow OA$	+
T17	$(u \rightarrow OA) \rightarrow OA$	+
T18	$OOA \rightarrow OA$	+
T19	$OA \leftrightarrow OOA$	-
T20	$(u \rightarrow OA) \leftrightarrow ((A \rightarrow Ou) \& (u \rightarrow OA))$	+
T21	$OA \leftrightarrow ((A \rightarrow Ou) \& (u \rightarrow OA))$	-
T22	OT	+
T23	$(T \rightarrow Ou) \& (u \rightarrow OT)$	+
T23'	$T \rightarrow Ou$	+
T23''	$u \rightarrow OT$	+
T23'''	$O(u \leftrightarrow T)$	+
T24	$A \rightarrow OA$	-
T25	$(A \rightarrow B) \rightarrow (A \rightarrow OB)$	-
T26	$(A \leftrightarrow B) \rightarrow ((A \rightarrow OB) \& (B \rightarrow OA))$	-
T27	$(\forall M)(\sim u \rightarrow O\sim M)$	+
T27'	$(\forall M)(\sim u \rightarrow OM)$	+
T28	$\sim u \rightarrow O\sim u$	+
T29	$\sim u \rightarrow Ou$	+
T30	$\sim u \rightarrow OF$	+
T31	$(\sim u \rightarrow OF) \& (F \rightarrow O\sim u)$	+
T31'	$O(\sim u \leftrightarrow F)$	+
T32	$\sim(u \rightarrow OF)$	+
T33	$\sim(u \rightarrow F)$	+
T34	$u \leftrightarrow T$	+
T35	$\sim u \leftrightarrow F$	+

The derivations of the theorems are easy. The refutations of the non-theorems are provided by the same matrices and lattice as in the proof of Observation 3. All

theorems of $\mathbf{KR}^{\delta+}$ receive the value 1 or 3, but each of the following expressions gets the value 0:

- | | | | |
|------|---|-----|--|
| T5a | $OO \leftrightarrow (O \rightarrow OO)$ | T21 | $O2 \leftrightarrow ((2 \rightarrow Ou) \& (u \rightarrow O2))$ |
| T12c | $O(1 \rightarrow 0) \leftrightarrow O\sim(1 \& \sim 0)$ | T24 | $1 \rightarrow O1$ |
| T13a | $(1 \rightarrow OO) \leftrightarrow \sim(1 \& \sim OO)$ | T25 | $(1 \rightarrow 1) \rightarrow (1 \rightarrow O1)$ |
| T19 | $O2 \leftrightarrow OO2$ | T26 | $(1 \leftrightarrow 1) \rightarrow ((1 \rightarrow O1) \& (1 \rightarrow O1))$ \square |

Observation 5. $\mathbf{R}^{\delta+}$ does not provide $OA \rightarrow \sim O\sim A$. $\mathbf{KR}^{\delta+}$ does.

Proof. Proof of negative claim: by MaGIC [19], which generates the following tables and lattice:

	\sim	O	\rightarrow	0	1	2	3	
0	3	0	0	3	3	3	3	\circ *3
1	2	2	1	0	2	2	3	\circ *2
*2	1	2	2	0	1	2	3	\circ 1
*3	0	3	3	0	0	0	3	\circ 0

$\&$ and \vee are interpreted as in the proof of Observation 3. $\mathbf{F} = 0$, $\mathbf{f} = 1$, $\mathbf{t} = 2$, $\mathbf{u} = \mathbf{T} = 3$. All theorems of $\mathbf{R}^{\delta+}$ receive the value 2 or 3, but $O1 \rightarrow \sim O\sim 1 = 1$. Proof of positive claim:

- | | | |
|---|--|----------------------------------|
| 1 | $OA \leftrightarrow (\sim O\mathbf{f} \rightarrow A)$ | See [9] and [10] |
| 2 | $(A \rightarrow B) \rightarrow ((A \rightarrow \sim B) \rightarrow (A \rightarrow (B \& \sim B)))$ | \mathbf{R} |
| 3 | $(A \rightarrow (B \& \sim B)) \rightarrow (A \rightarrow (\mathbf{u} \rightarrow \sim \mathbf{u}))$ | \mathbf{KR} |
| 4 | $OA \rightarrow (O\sim A \rightarrow O(\mathbf{u} \rightarrow \sim \mathbf{u}))$ | 1 2 3 |
| 5 | $\sim O(\mathbf{u} \rightarrow \sim \mathbf{u})$ | $D3b \rightarrow D5 \rightarrow$ |
| 6 | $OA \rightarrow \sim O\sim A$ | 4 5 \mathbf{R} \square |

Observation 5 is interesting because $OA \rightarrow \sim O\sim A$ is the relevant version of $OA \supset \sim O\sim A$. This theorem is often regarded as characteristic for deontic logic. There can be no doubt that Mally accepted it, even though he did not endorse it explicitly. His approval becomes clear from his discussion of the logic of willing (bouletic or boulomaic logic), which he saw as formally similar to deontic logic, but in which OA is read as “it is correctly desired that A .” In this context, he gave the following paraphrase of D5: “somebody who correctly desires that A does not desire that not A ; (not even implicitly); correct desire is free of contradiction” (*Wer richtig will, will nicht (auch nicht impliziterweise) das Negat des Gewollten; richtiges Wollen ist widerspruchsfrei*, [14, ch. 4, §10, p. 49, ad (V)]). This is a paraphrase of $OA \supset \sim O\sim A$ rather than D5.

4 Relevant Strict Implication

Observation 6. \mathbf{KR}^{δ} has the following theorems:

- (a) $A \rightarrow O\sim O\sim A$
- (b) $OOA \rightarrow A$

(c) $O(OA \vee B) \rightarrow (A \vee OB)$

Proof. (a) follows from $D1^\rightarrow$, $D3a^\rightarrow$ and $D3b^\rightarrow$: see [10]. (b) follows from $OA \rightarrow \sim O\sim A$ (Observation 5). (c) follows from theorem

KR2 $(A \& ((B \circ C) \circ D)) \rightarrow (((A \circ C) \& (B \circ D)) \circ C)$,

where $A \circ B =_{\text{df}} \sim(A \rightarrow \sim B)$, which is proven in [2, §54.1, pp. 263–4]. The details are as follows:

- | | | |
|---|--|---|
| 1 | $(C \rightarrow ((C \rightarrow A) \vee (D \rightarrow B))) \rightarrow (A \vee (D \rightarrow (C \rightarrow B)))$ | KR2 Df \circ R |
| 2 | $(\sim O\mathbf{f} \rightarrow ((\sim O\mathbf{f} \rightarrow A) \vee (\mathbf{t} \rightarrow B))) \rightarrow (A \vee (\mathbf{t} \rightarrow (\sim O\mathbf{f} \rightarrow B)))$ | 1 |
| 3 | $OA \leftrightarrow (\sim O\mathbf{f} \rightarrow A)$ | [9] [10] |
| 4 | $O(OA \vee B) \rightarrow (A \vee OB)$ | 2 3 R □ |

Because (a), (b) and (c) are unacceptable from a modern point of view (cf. [7]), **KR $^\delta$** is not acceptable either. This shortcoming can be removed as follows.

Definition 8. **KR O** consists of **KR** plus the following axioms:

- O1** $(A \rightarrow B) \rightarrow (OA \rightarrow OB)$
- O2** $O(OA \rightarrow A)$
- O3** $O\mathbf{u}$
- O4** $OA \rightarrow \sim O\sim A$

Observation 7. **KR O** has the same theorems as **KR $^\delta$** .

Proof. O1 and O2 follow from $D1^\rightarrow$, $D3a^\rightarrow$ and $D3b^\rightarrow$: see [12]. O3 follows from $D4^\rightarrow$. O4 follows from Observation 5. $D1^\rightarrow$, $D2^\rightarrow$, $D3a^\rightarrow$ and $D3b^\rightarrow$ follow from O1 and O2: see [10] and [12]. $D4^\rightarrow$ follows from O3. $D5^\rightarrow$ follows from O1–O4 because:

- | | | |
|---|--|---|
| 1 | $O\mathbf{u}$ | O3 |
| 2 | $\mathbf{u} \rightarrow \sim(\mathbf{u} \rightarrow \sim\mathbf{u})$ | R |
| 3 | $O\sim(\mathbf{u} \rightarrow \sim\mathbf{u})$ | 1 2 O1 |
| 4 | $\sim O(\mathbf{u} \rightarrow \sim\mathbf{u})$ | 3 O4 |
| 5 | $\sim(\mathbf{u} \rightarrow O\sim\mathbf{u})$ | 4 D3a $^\rightarrow$ □ |

Definition 9. Relevant alethic modal system **KR $^\square$** has the following axioms and rules in addition to those of **KR**:

- K** $\square(A \rightarrow B) \rightarrow (\square A \rightarrow \square B)$
- C** $(\square A \& \square B) \rightarrow \square(A \& B)$
- T** $\square A \rightarrow A$
- 4** $\square A \rightarrow \square\square A$
- Nec** From A to infer $\square A$.

This system is the same as **R $^\square$** [1, §27.1.3, pp. 343–4] plus $(A \& \sim A) \rightarrow B$.

Definition 10. $A \Rightarrow B =_{df} \Box(A \rightarrow B)$

The symbol \Rightarrow represents *relevant strict implication*, which is stronger than classical strict implication and relevant implication in the sense that both $(A \Rightarrow B) \rightarrow (A \rightarrow B)$ and $(A \Rightarrow B) \rightarrow (A \rightarrow B)$ are theorems, where $A \rightarrow B =_{df} \Box(A \supset B)$ and $A \supset B =_{df} \sim A \vee B$.

Definition 11. *Mixed alethic-deontic system $\mathbf{KR}^{\Box O}$ has the following axioms in addition to those of \mathbf{KR}^{\Box} :*

- O1'** $(A \Rightarrow B) \rightarrow (OA \rightarrow OB)$
- O2–O4** *As above (Definition 8)*

Observation 8. $\mathbf{KR}^{\Box O}$ does not provide the formulas mentioned in Observations 3 and 6.

Proof. By MaGIC [19]. The details are as follows.

1. Refutation of the formulas mentioned in Observation 3: identical with the proof of Observation 3, but add:

$$\begin{array}{c|cccc} A & 0 & *1 & 2 & *3 \\ \hline \Box A & 0 & 1 & 2 & 3 \end{array}$$

2. Refutation of the formulas mentioned in Observation 6: by the following tables and lattice, in which 3 is the only designated value:

	\sim	\Box	O
0	3	0	0
1	2	0	3
2	1	0	0
*3	0	3	3

	\rightarrow	0	1	2	3
0	3	3	3	3	3
1	2	3	2	3	3
2	1	1	3	3	3
3	0	1	2	3	3

$F = f = 0, t = T = u = 3$. All theorems get the value 3, but $2 \rightarrow O \sim O \sim 2 = 1, OO1 \rightarrow 1 = 1$ and $O(O1 \vee 2) \rightarrow (1 \vee O2) = 1$. □

Observation 8 implies that we have to make only one change in \mathbf{KR}^O (replace O1 by O1') to obtain a system that is not only acceptable in light of Menger's and Mally's criteria, but also from a modern point of view.

Føllesdal and Hilpinen have suggested that Mally's system can be repaired by replacing axiom D1 by the weaker D1':

- D1** $((A \supset OB) \& (B \supset C)) \supset (A \supset OC)$
- D1'** $((A \supset OB) \& (B \rightarrow C)) \supset (A \supset OC),$

where \rightarrow is "some sort of strict implication," for example strict implication as defined in modal system **S4** (see [6, pp. 5–6]). This replacement of D1 by D1' is similar to the replacement of O1 by O1'.

5 Historical Digression

Some of the observations we have made could already have been made in 1928, right after Ivan Orlov had made the first contribution to relevant logic (see [17]; on Orlov, see [3]). Orlov's system **OR** has the same theorems as the implication-negation fragment of **R** (see [5]). His relevant modal system **OR**[□] (called **ORS4** in [5]) consists of **OR** plus axioms □K, □T, □4 and rule Nec (as in Definition 9). We define **OR**^O as **OR** plus axioms O1–O4 (as in Definition 8) and **OR**^{□O} as **OR**[□] plus axioms O1'–O4 (as in Definition 11). Neither **OR**^O nor **OR**^{□O} provides $A \rightarrow OA$ or $OA \rightarrow A$ (by Observation 3). **OR**^O provides the dubious theorems $A \rightarrow O\sim O\sim A$ and $OOA \rightarrow A$ (proof: identical with the proof of Observation 6, but use O4 instead of Observation 5). **OR**^{□O} does not provide these theorems (by Observation 8).

6 Conclusion

Mally formulated the first system of deontic logic, but his system was unacceptable (Section 1). Where did he go wrong? It can be argued that it was his adoption of classical propositional logic that led him into trouble. Adopting some system of strict implication does not help as long as one uses one of the Lewis systems (Section 2). Adopting a system of relevance logic does; system **KR** is particularly suitable (Section 3). The resulting relevant deontic system expresses Mally's assumptions while it avoids both Menger's and Mally's own objections. It is not satisfactory from a modern point of view, but it can be reformulated in such a way that only one axiom needs modification to make it acceptable (Section 4).

References

1. Anderson, A.R., Belnap Jr., N.D.: *Entailment: The Logic of Relevance and Necessity*, vol. 1. Princeton University Press, Princeton (1975)
2. Anderson, A.R., Belnap Jr., N.D., Dunn, J.M.: *Entailment: The Logic of Relevance and Necessity*, vol. 2. Princeton University Press, Princeton (1992)
3. Bazhanov, V.A.: The scholar and the 'Wolfhound Era': The fate of Ivan E. Orlov's ideas in logic, philosophy, and science. *Science in Context* 16, 535–550 (2003)
4. Chellas, B.F., Segerberg, K.: Modal logics in the vicinity of S1. *Notre Dame Journal of Formal Logic* 37, 1–24 (1996)
5. Došen, K.: The first axiomatization of relevant logic. *Journal of Philosophical Logic* 21, 339–356 (1992)
6. Føllesdal, D., Hilpinen, R.: Deontic logic: An introduction. In: Hilpinen, R. (ed.) *Deontic Logic: Introductory and Systematic Readings*, 2nd edn., pp. 1–35. Reidel, Dordrecht (1981)
7. Goble, L.: The Andersonian reduction and relevant deontic logic. In: Brown, B., Woods, J. (eds.) *New Studies in Exact Philosophy: Logic, Mathematics and Science*. Proceedings of the 1999 Conference of the Society of Exact Philosophy, pp. 213–246. Hermes Science Publications, Paris (2001)

8. Lemmon, E.J.: New foundations for Lewis modal systems. *The Journal of Symbolic Logic* 22, 176–186 (1957)
9. Lokhorst, G.J.C.: Anderson’s relevant deontic and eubouliatic systems. *Notre Dame Journal of Formal Logic* 49, 65–73 (2008)
10. Lokhorst, G.J.C.: Andersonian deontic logic, propositional quantification, and Mally. *Notre Dame Journal of Formal Logic* 47, 385–395 (2006)
11. Lokhorst, G.J.C.: Mally’s deontic logic. In: Zalta, E.N. (ed.) *The Stanford Encyclopedia of Philosophy* (2002), <http://plato.stanford.edu/entries/mally-deontic/>
12. Lokhorst, G.J.C., Goble, L.: Mally’s deontic logic. *Grazer philosophische Studien* 67, 37–57 (2004)
13. McCune, W.: Prover9 version 2008-11A (November 2008), <http://www.cs.unm.edu/~mccune/prover9/>
14. Mally, E.: *Grundgesetze des Sollens: Elemente der Logik des Willens*. Leuschner und Lubensky, Graz (1926); Reprinted in Mally, E.: *Logische Schriften: Großes Logikfragment, Grundgesetze des Sollens*. In: Wolf, K., Weingartner, P. (eds.), pp. 227–324. Reidel, Dordrecht (1971)
15. Menger, K.: A logic of the doubtful: On optative and imperative logic. In: *Reports of a Mathematical Colloquium. 2nd Series, vol. (2)*, pp. 53–64. Indiana University Press, Notre Dame (1939)
16. Morscher, E.: Mallys Axiomensystem für die deontische Logik: Rekonstruktion und kritische Würdigung. In: Hieke, A. (ed.) *Ernst Mally: Versuch einer Neubewertung*, pp. 81–165. Academia Verlag, Sankt Augustin (1998)
17. Orlov, I.E.: *Ischislenie sovместimosti predlozhenij*. *Matematicheskij sbornik* 35, 263–286 (1928), <http://mi.mathnet.ru/msb/v35/i3/p263>
18. Rabe, F., Pudlák, P., Sutcliffe, G., Shen, W.: Solving the \$100 modal logic challenge. *Journal of Applied Logic* 7, 113–130 (2009)
19. Slaney, J.K.: MaGIC (Matrix Generator for Implication Connectives) version 2.2.1 (November 2008), <http://users.rsise.anu.edu.au/~jks/magic.html>
20. Zeman, J.J.: *Modal Logic: The Lewis-Modal Systems*. The Clarendon Press, Oxford (1973), <http://www.clas.ufl.edu/users/jzeman/modallogic/>

Appendix

Proof of $A \varepsilon \exists OA$ generated by Prover9 [13]. The input is on lines 1–14, the output on lines 15–50. The methodology is explained in [18]. For readability, the formulas were converted from prefix notation to infix notation by a Perl script. As in [4], RRTE refers to the rule of replacement of tautological equivalents.

1	$\vdash (A \rightarrow OA) \& (OA \rightarrow A)$	goal
2	$\not\vdash A$ or $\not\vdash B$ or $\vdash A \& B$	MP PL
3	$\not\vdash A$ or $\not\vdash A \rightarrow B$ or $\vdash B$	SMP
4	$\vdash (A \& B) \rightarrow A$	□PL
5	$\vdash (A \& (A \supset B)) \rightarrow B$	□PL
6	$\not\vdash F$ or $\vdash F^{\sim\sim A} / A$	RRTE PL
7	$\not\vdash F$ or $\vdash F^{(\sim A \supset \sim B)} / (B \supset A)$	RRTE PL
8	$\not\vdash F$ or $\vdash F^{((A \& B) \supset C)} / (A \supset (B \supset C))$	RRTE PL

9	$\not\vdash F$ or $\vdash F^{(\sim A \supset A)} /_A$	RRTE PL
10	$\not\vdash F$ or $\vdash F^{\sim(A \& \sim B)} /_{(A \supset B)}$	RRTE PL
11	$\not\vdash F$ or $\vdash F^{\Box(A \supset B)} /_{(A \supset B)}$	Def \supset
12	$\vdash (A \supset OB) \supset O(A \supset B)$	D3a ³
13	$\vdash O(A \supset B) \supset (A \supset OB)$	D3b ³
14	$\vdash \sim(\mathbf{u} \supset O\sim\mathbf{u})$	D5 ³
15	$\not\vdash (a \supset Oa) \& (Oa \supset a)$	deny 1
16	$\not\vdash F$ or $\vdash F^{(A \supset \sim B)} /_{(B \supset \sim A)}$	para 6 7
17	$\not\vdash F$ or $\vdash F^{(\sim A \supset B)} /_{(\sim B \supset A)}$	para 6 7
18	$\not\vdash F$ or $\vdash F^{(A \supset \sim A)} /_{\sim A}$	para 6 9
19	$\not\vdash F$ or $\vdash F^{\sim(A \supset B)} /_{(A \& \sim B)}$	para 10 6
20	$\not\vdash F$ or $\vdash F^{\sim(A \& B)} /_{(A \supset \sim B)}$	para 6 10
21	$\not\vdash F$ or $\vdash F^{(\sim A \supset \sim B)} /_{(B \supset A)}$	para 7 11, rewrite 11, flip
22	$\not\vdash F$ or $\vdash F^{((A \& B) \supset C)} /_{(A \supset (B \supset C))}$	para 8 11, rewrite 11, flip
23	$\vdash O((OA \& B) \supset A)$	hyper 3 4 12
24	$\not\vdash F$ or $\vdash F^{(A \supset \sim A)} /_{\Box \sim A}$	para 18 11, flip
25	$\vdash O\Box \sim A \supset (A \supset O\sim A)$	para 24 13
26	$\not\vdash F$ or $\vdash F^{(\sim A \supset B)} /_{(\sim B \supset A)}$	para 17 11, rewrite 11
27	$\not\vdash F$ or $\vdash F^{(A \& A)} /_A$	para 18 19, rewrite 6 6, flip
28	$\not\vdash F$ or $\vdash F^{(A \& B)} /_{(B \& A)}$	para 16 19, rewrite 19 6 6
29	$\vdash O(OA \supset A)$	para 27 23
30	$\vdash O((A \& OB) \supset B)$	para 28 23
31	$\not\vdash \sim A$ or $\not\vdash B \supset A$ or $\vdash \sim B$	para 21 3
32	$\vdash \sim A \supset (A \supset \sim B)$	para 21 4, rewrite 20
33	$\vdash \sim A \supset (A \supset B)$	para 6 32
34	$\vdash (A \& \sim A) \supset B$	para 6 33, rewrite 22
35	$\vdash O((A \& \sim A) \supset B)$	hyper 3 34 12
36	$\vdash O(A \supset A)$	para 24 35, rewrite 20 6 11
37	$\vdash A \supset OA$	hyper 3 36 13
38	$\not\vdash Oa \supset a$	ur 2 37 15
39	$\not\vdash A \& (A \supset (Oa \supset a))$	ur 3 5 38
40	$\vdash O(A \supset (OB \supset B))$	para 22 30
41	$\vdash O((\sim A \& OA) \supset B)$	para 26 40, rewrite 19 28
42	$\vdash (\sim A \& OA) \supset OB$	hyper 3 41 13
43	$\vdash \sim OA \supset (OB \supset B)$	para 21 42, rewrite 20 16 6
44	$\vdash (\sim OA \& OB) \supset B$	para 22 43
45	$\vdash (OA \& \sim OB) \supset A$	para 28 44
46	$\vdash OA \supset (\sim OB \supset A)$	para 22 45
47	$\vdash \sim OA \supset (OB \supset B)$	hyper 3 29 46
48	$\not\vdash \sim OA$	ur 2 47 39
49	$\vdash \sim O\Box \sim \mathbf{u}$	hyper 31 14 25
50	\perp	resolve 49 48